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TRANSIENT CHARACTERISTICS OF A ROTATING PLASMA

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CONTENTS

TABLES

FIGURES

TRANSIENT CHARACTERISTICS OF A ROTATING PLASMA*

Ching-Sheng Wu

I. INTRODUCTION

This Report concerns the rotational motion of plasma between two concentric cylindrical electrodes. The azimuthal rotation of the plasma is generated by the applied magnetic field, which is in the direction parallel to the axis of the electrodes (see Fig. 1). The cylindrical electrodes are assumed to be very long, and the radial current is uniformly distributed and axially symmetric. Before the magnetic field is imposed, the electrical current flows in the radial direction and the plasma is motionless. When the uniform magnetic field is applied impulsively, interaction between the current and the field takes place immediately so that the plasma is accelerated in the azimuthal direction. The plasma motion is expected to reach a steady state at a later time, when the ponderomotive force and viscous force become the same order of magnitude.

From the microscopic point of view, the problem may be very complicated, especially in the region near the electrode. In an attempt to simplify the physical model, it is assumed that plasma density is high, so that it may be treated as a continuous medium. It is also postulated that the electron cyclotron frequency is small compared to the mean collision frequency. Under these conditions, we may consider the transport coefficients of the plasma as essentially scalar quantities.

The stationary solution of this problem is first obtained by Gordeev (Ref. 1). The purpose of the following discussion is to obtain the transient solution of the velocity distribution so that the time interval required to reach the stationary state may be estimated.

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II. METHOD OF SOLUTION OF VELOCITY DISTRIBUTION

Since, in the present problem, the radial current density takes the form $J_r = I/(2\pi r)$, the hydrodynamic equation of motion may be written as

$$\frac{\partial v_\theta}{\partial t} - \nu \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \right) = - \frac{IB_0}{2\pi r \rho} \tag{1}$$

where I is the total current per unit length and B_0 is the constant longitudinal magnetic field. The initial and boundary conditions may be stated as follows:

$v_\theta = 0$	everywhere	at $t = 0$
$v_\theta = 0$	at $r = r_1$ and $r = r_2$	for all time

In the following discussion, ν and ρ are assumed to be the averaged values of kinematic viscosity and density of the plasma.

In an attempt to solve the present problem, the Laplace transform

$$\overline{v_\theta} = \int_0^\infty v_\theta e^{-st}\, dt$$

is introduced. Corresponding to this transformation, Eq. (1) becomes

$$s\overline{v_\theta} - \nu \left(\frac{\partial^2 \overline{v_\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v_\theta}}{\partial r} - \frac{\overline{v_\theta}}{r^2} \right) = - \frac{IB_0}{2\pi \rho r s}$$

or

$$\frac{\partial^2 \overline{v_\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v_\theta}}{\partial r} - \left(\frac{s}{\nu} + \frac{1}{r^2} \right) \overline{v_\theta} = \frac{IB_0}{2\pi \rho r s \nu} \tag{2}$$

Let $s/\nu = \lambda^2$ and $\eta = \lambda r$; then Eq. (2) may be written as

$$\frac{\partial^2 \overline{v_\theta}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \overline{v_\theta}}{\partial \eta} - \left(1 + \frac{1}{\eta}\right)\overline{v_\theta} = \frac{IB_0}{2\pi\rho\nu^2\eta\lambda^3} \tag{3}$$

The boundary conditions of $\overline{v_\theta}$ are

$$\begin{aligned} \overline{v_\theta} = 0 \qquad \text{at } \eta &= \eta_1 = \lambda r_1 \\ \eta &= \eta_2 = \lambda r_2 \end{aligned} \tag{4}$$

The homogeneous solutions of Eq. (3) are

$$\overline{v_\theta} = I_1(\eta) \qquad \text{and } \overline{v_\theta} = K_1(\eta)$$

The Wronskian of $I_1(\eta)$ and $K_1(\eta)$ (Ref. 2) is

$$I_1(\eta)\, K_1'(\eta) - I_1'(\eta)\, K_1(\eta) = -\frac{1}{\eta}$$

Therefore, the complete solution may immediately be found as follows:

$$\overline{v_\theta} = I_1(\eta) \int \frac{K_1(\eta)}{\frac{1}{\eta}} \left(\frac{IB_0}{2\pi\rho\nu^2\eta\lambda^3}\right) d\eta - K_1(\eta) \int \frac{I_1(\eta)}{\frac{1}{\eta}} \left(\frac{IB_0}{2\pi\rho\nu^2\eta\lambda^3}\right) d\eta$$

$$= \left[-I_1(\eta)\, K_0(\eta) - K_1(\eta)\, I_0(\eta) + AI_1(\eta) + BK_1(\eta)\right] \frac{IB_0}{2\pi\rho\nu^2\lambda^3}$$

$$= \left[-\frac{1}{\eta} + AI_1(\eta) + BK_1(\eta)\right] \frac{IB_0}{2\pi\rho\nu^2\lambda^3} \tag{5}$$

where A and B are two arbitrary constants which may be determined by the boundary conditions (Eq. 4). Consequently, $\overline{v_\theta}$ may be expressed in terms of r:

$$\overline{v_\theta} = \frac{IB_0}{2\pi\rho\nu^2\lambda^3}\left\{-\frac{1}{\lambda r} + \frac{1}{I_1(\lambda r_1)\,K_1(\lambda r_2) - I_1(\lambda r_2)\,K_1(\lambda r_1)}\left[\left(\frac{K_1(\lambda r_2)}{\lambda r_1} - \frac{K_1(\lambda r_1)}{\lambda r_2}\right) I_1(\lambda r) - \left(\frac{I_1(\lambda r_2)}{\lambda r_1} - \frac{I_1(\lambda r_1)}{\lambda r_2}\right) K_1(\lambda r)\right]\right\} \tag{6}$$

To find the inverse transform of $\overline{v_\theta}$, the inversion integral (Ref. 3) is performed.

$$v_\theta = -\frac{IB_0\,t}{2\,\rho r} + \frac{IB_0}{4\pi^2\rho\nu^2 i}\int_{a-i\infty}^{a+i\infty} ds\left\{\frac{1}{I_1(\lambda r_1)\,K_1(\lambda r_2) - I_1(\lambda r_2)\,K_1(\lambda r_1)} \times \left[\left(\frac{K_1(\lambda r_2)}{\lambda^4 r_1} - \frac{K_1(\lambda r_1)}{\lambda^4 r_2}\right) I_1(\lambda r) - \left(\frac{I_1(\lambda r_2)}{\lambda^4 r_1} - \frac{I_1(\lambda r_1)}{\lambda^4 r_2}\right) K_1(\lambda r)\right]\right\} e^{st} \tag{7}$$

By analyzing the asymptotic behavior of the integrand for small λ, one may show that the function is not multi-valued and $s = 0$ is a pole of second order. If a Bromwich contour is used, the above integral may readily be evaluated if the residues of all poles are known. Since there is no branch point in the complex s-plane, the contour shown in Fig. 2 may be adopted. The poles are situated on the negative real axis. Thus,

$$v_\theta = \frac{IB_0}{2\pi\rho\nu^2}\left\{-\frac{t\nu^2}{r} + R_0 + \sum_{n=1} R_n\right\}$$

where R_0 is the residue of the second-order pole $s = 0$ and R_n are the residues of the simple poles $s = s_n$. Since

$$R_0 = \frac{t\nu^2}{r} + \frac{\nu}{2}\left\{ r \ln\left(\frac{r}{r_1}\right) - r\left(1 - \frac{r_1^2}{r^2}\right)\frac{\ln\left(\frac{r_2}{r_1}\right)}{(r_2^2 - r_1^2)}\right\}^{*}$$

$$R_n = \frac{2\nu I_1(\lambda_n r_1)\, I_1(\lambda_n r_2)}{\lambda_n^2\,[I_1^2\,(\lambda_n r_2) - I_1^2(\lambda_n r_1)]}\left\{\left[\frac{K_1(\lambda_n r_2)}{r_1} - \frac{K_1(\lambda_n r_1)}{r_2}\right] I_1(\lambda_n r)\right.$$

$$\left. - \left[\frac{I_1(\lambda_n r_2)}{r_1} - \frac{I_1(\lambda_n r_1)}{r_2}\right] K_1(\lambda_n r)\right\} e^{\nu\lambda_n^2 t}\,{}^{*}$$

Therefore,

$$v_\theta = \frac{IB_0}{4\pi\rho\nu}\left\{ r \ln\left(\frac{r}{r_1}\right) - r\left(1 - \frac{r_1^2}{r^2}\right)\frac{\ln\left(\frac{r_2}{r_1}\right)}{\left(1 - \frac{r_1^2}{r_2^2}\right)}\right.$$

$$+ \sum_{n=1}^{\infty} \frac{4 I_1(\lambda_n r_1)\, I_1(\lambda_n r_2)}{\lambda_n^2\,[I_1^2\,(\lambda_n r_2) - I_1^2(\lambda_n r_1)]}\left[\left(\frac{K_1(\lambda_n r_2)}{r_1} - \frac{K_1(\lambda_n r_1)}{r_2}\right) I_1(\lambda_n r)\right.$$

$$\left.\left. - \left(\frac{I_1(\lambda_n r_2)}{r_1} - \frac{I_1(\lambda_n r_1)}{r_2}\right) K_1(\lambda_n r)\right] e^{\nu\lambda_n^2 t}\right\} \tag{8}$$

It is seen that since $\lambda_n = s_n/\nu$ and s_n's have negative values, it is more convenient to introduce a variable α_n, such that

$$\lambda_n = \alpha_n i$$

*The calculation is straightforward and may be found in the Appendixes.

Again, according to Ref. 2,

$$K_1(\alpha_n i) = \frac{1}{2}\pi\left[-J_1(\alpha_n) + iY_1(\alpha_n)\right]$$

$$I_1(\alpha_n i) = iJ_1(\alpha_n)$$

One may write:

$$\left[\frac{K_1(\lambda_n r_2)}{r_1} - \frac{K_1(\lambda_n r_1)}{r_2}\right] I_1(\lambda_n r) - \left[\frac{I_1(\lambda_n r_2)}{r_1} - \frac{I_1(\lambda_n r_1)}{r_2}\right] K_1(\lambda_n r)$$

$$= -\frac{1}{2}\pi\left[\left(\frac{Y_1(\alpha_n r_2)}{r_1} - \frac{Y_1(\alpha_n r_1)}{r_2}\right) J_1(\alpha_n r) - \left(\frac{J_1(\alpha_n r_2)}{r_1} - \frac{J_1(\alpha_n r_1)}{r_2}\right) Y_1(\alpha_n r)\right]$$

Therefore, Eq. (8) may be rewritten in the form

$$v_\theta = -\frac{IB_0}{4\pi\mu}\left\{ r\left[1 - \frac{r_1^2}{r^2}\right]\frac{\ln\left(\frac{r_2}{r_1}\right)}{1 - \frac{r_1^2}{r_2^2}} - r\ln\left(\frac{r}{r_1}\right)\right.$$

$$+ \sum_{n=1}^{\infty}\frac{2\pi J_1(\alpha_n r_2)\, J_1(\alpha_n r_1)}{\alpha_n^2\left[J_1^2(\alpha_n r_2) - J_1^2(\alpha_n r_1)\right]}\left[\left(\frac{Y_1(\alpha_n r_1)}{r_2} - \frac{Y_1(\alpha_n r_2)}{r_1}\right) J_1(\alpha_n r)\right.$$

$$\left.\left. - \left(\frac{J_1(\alpha_n r_1)}{r_2} - \frac{J_1(\alpha_n r_2)}{r_1}\right) Y_1(\alpha_n r)\right] e^{-\nu\alpha_n^2 t}\right\} \qquad (9)$$

III. CONCLUSIONS

From the result of Eq. (9), a few remarks can be made immediately. First, the velocity is linearly proportional to IB_0 but inversely proportional to v_θ. In other words, in the case of two different kinds of plasma, the one with higher viscosity will have lower rotational velocity. This is physically conceivable. Second, the viscous dissipation is again inversely proportional to μ. Besides these two points, it is also seen that the transient time interval is inversely proportional to the kinematic viscosity.

Numerical calculations of the velocity distribution, total kinematic energy and total viscous dissipation per unit length of the cylindrical space have been done for three different cases. The results are tabulated and plotted in Tables 1 and 2 and Fig. 3, 4, and 5.

Table 1. Variations of velocity distribution of plasma
(1) $r_1 = 1$, $r_2 = 8$

$R_n = r_1 + \frac{n(r_2 - r_1)}{10}$	$v_\theta \left(\frac{IB_0}{4\pi\rho\nu}\right)^{-1}$									
	$\nu t = 0$	0.5	1.0	1.5	2.0	3.0	5.0	8.0	12.0	15.0
R_0	0	0	0	0	0	0	0	0	0	0
R_1	0	0.0607	0.0909	0.1119	0.1281	0.1526	0.1839	0.2084	0.2222	0.2264
R_2	0	0.0606	0.1037	0.1362	0.1623	0.2025	0.2547	0.2958	0.3189	0.3264
R_3	0	0.0506	0.0949	0.1317	0.1626	0.2119	0.2773	0.3293	0.3585	0.3675
R_4	0	0.0418	0.0819	0.1181	0.1499	0.2022	0.2734	0.3306	0.3629	0.3728
R_5	0	0.0353	0.0702	0.1031	0.1329	0.1832	0.2531	0.3100	0.3422	0.3521
R_6	0	0.0305	0.0605	0.0887	0.1146	0.1588	0.2214	0.2729	0.3021	0.3111
R_7	0	0.0267	0.0517	0.0744	0.0949	0.1301	0.1804	0.2222	0.2460	0.2533
R_8	0	0.0228	0.0415	0.0574	0.0716	0.0957	0.1305	0.1595	0.1761	0.1812
R_9	0	0.0159	0.0260	0.0342	0.0413	0.0534	0.0708	0.0854	0.0938	0.0963
R_{10}	0	0	0	0	0	0	0	0	0	0

Table 1 (Cont'd)

(2) $r_1 = 1, \quad r_2 = 9$

$R_n = r_1 + \frac{n(r_2 - r_1)}{10}$	$v_\theta \left(\frac{IB_0}{4\pi\rho\nu}\right)^{-1}$									
	$\nu t = 0$	0.5	1.0	1.5	2.0	3.0	5.0	8.0	12.0	15.0
R_0	0	0	0	0	0	0	0	0	0	0
R_1	0	0.0624	0.0954	0.1185	0.1366	0.1644	0.2007	0.2327	0.2541	0.2621
R_2	0	0.0579	0.1023	0.1367	0.1647	0.2089	0.2695	0.3229	0.3586	0.3720
R_3	0	0.0465	0.0893	0.1264	0.1585	0.2114	0.2864	0.3537	0.3990	0.4160
R_4	0	0.0378	0.0750	0.1099	0.1417	0.1965	0.2772	0.3510	0.4009	0.4197
R_5	0	0.0318	0.0635	0.0943	0.1233	0.1749	0.2534	0.3264	0.3763	0.3950
R_6	0	0.0274	0.0546	0.0809	0.1059	0.1507	0.2202	0.2860	0.3312	0.3483
R_7	0	0.0240	0.0471	0.0685	0.0884	0.1239	0.1795	0.2327	0.2695	0.2834
R_8	0	0.0208	0.0386	0.0541	0.0680	0.0924	0.1306	0.1675	0.1931	0.2027
R_9	0	0.0151	0.0251	0.0332	0.0403	0.0526	0.0716	0.0902	0.1031	0.1079
R_{10}	0	0	0	0	0	0	0	0	0	0
	(3) $r_1 = 1, \quad r_2 = 10$									
R_0	0	0	0	0	0	0	0	0	0	0
R_1	0	0.0633	0.0988	0.1239	0.1435	0.1737	0.2148	0.2530	0.2816	0.2938
R_2	0	0.0550	0.0998	0.1355	0.1651	0.2125	0.2795	0.3428	0.3906	0.4110
R_3	0	0.0429	0.0837	0.1204	0.1529	0.2079	0.2897	0.3691	0.4295	0.4555
R_4	0	0.0346	0.0689	0.1019	0.1330	0.1883	0.2749	0.3614	0.4280	0.4567
R_5	0	0.0289	0.0518	0.0863	0.1140	0.1650	0.2479	0.3331	0.3994	0.4280
R_6	0	0.0248	0.0496	0.0740	0.0976	0.1414	0.2141	0.2904	0.3504	0.3765
R_7	0	0.0217	0.0430	0.0632	0.0822	0.1170	0.1747	0.2361	0.2849	0.3061
R_8	0	0.0190	0.0360	0.0509	0.0645	0.0886	0.1282	0.1705	0.2044	0.2192
R_9	0	0.0143	0.0242	0.0323	0.0393	0.0515	0.0713	0.0925	0.1095	0.1170
R_{10}	0	0	0	0	0	0	0	0	0	0

Table 2. Variations of total kinetic energy and viscous dissipation per unit length

	(1) $r_1 = 1, \quad r_2 = 8$		(2) $r_1 = 1, \quad r_2 = 9$		(3) $r_1 = 1, \quad r_2 = 10$	
νt	KE^*	Φ^{**}	KE^*	Φ^{**}	KE^*	Φ^{**}
0	0	0	0	0	0	0
0.5	0.0176	0.0090	0.0193	0.0079	0.0209	0.0069
1.0	0.0587	0.0259	0.0656	0.0234	0.0719	0.0212
1.5	0.1140	0.0465	0.1300	0.0424	0.1442	0.0389
2.0	0.1776	0.0694	0.2066	0.0638	0.2323	0.0588
3.0	0.3142	0.1177	0.3803	0.1108	0.4394	0.1035
5.0	0.5678	0.2074	0.7398	0.2072	0.9018	0.2008
8.0	0.8296	0.3005	1.1815	0.3262	1.5452	0.3363
12.0	1.0001	0.3613	1.5432	0.4241	2.1674	0.4683
15.0	1.0557	0.3812	1.6998	0.4645	2.4694	0.5326

$$^{*}KE = 8\,\rho \left(\frac{IB_0}{\nu}\right)^{-2} \int \left(\frac{1}{2}\,\rho v_\theta^2\right) r\,dr$$

$$^{**}\Phi = \frac{8\,\mu}{I^2 B_0^2} \int \mu \left(\frac{\partial v_\theta}{\partial r}\right)^2 r\,dr$$

Finally, it must be remarked that the present solution provides a good approximation only when a longitudinal pressure gradient is imposed previously such that the longitudinal motion of the plasma becomes negligible.

APPENDIX A

Evaluation of Residue of the Second-Order Pole $s = 0$

If the integrand is denoted by f, i.e.,

$$f = \frac{1}{I_1(\lambda r_1)\,K_1(\lambda r_1) - I_1(\lambda r_2)\,K_1(\lambda r_1)} \left\{ \left[\frac{K_1(\lambda r_2)}{\lambda^4 r_1} - \frac{K_1(\lambda r_1)}{\lambda^4 r_2} \right] I_1(\lambda r) - \left[\frac{I_1(\lambda r_2)}{\lambda^4 r_1} - \frac{I_1(\lambda r_1)}{\lambda^4 r_2} \right] K_1(\lambda r) \right\} e^{st} \tag{A-1}$$

the residue R_0 may be evaluated by using the relation

$$R_0 = \left(\frac{d}{ds}\, s^2 f \right)_{s=0} = \left(\frac{\nu}{2\lambda} \frac{d}{d\lambda}\, \lambda^4 f \right)_{\lambda=0}$$

$$= \frac{\nu}{2\lambda} \frac{d}{d\lambda} \left\{ \frac{1}{I_1(\lambda r_1)\,K_1(\lambda r_2) - I_1(\lambda r_2)\,K_1(\lambda r_1)} \left[\left(\frac{K_1(\lambda r_2)}{r_1} - \frac{K_1(\lambda r_1)}{r_2} \right) I_1(\lambda r) \right.\right.$$

$$\left.\left. - \left(\frac{I_1(\lambda r_2)}{r_1} - \frac{I_1(\lambda r_1)}{r_2} \right) K_1(\lambda r) \right] e^{st} \right\}_{\lambda=0} \tag{A-2}$$

If

$$p = \left\{ \left[\frac{K_1(\lambda r_2)}{r_1} - \frac{K_1(\lambda r_1)}{r_2} \right] I_1(\lambda r) - \left[\frac{I_1(\lambda r_2)}{r_1} - \frac{I_1(\lambda r_1)}{r_2} \right] K_1(\lambda r) \right\} e^{st}$$

$$q = \left[I_1(\lambda r_1)\,K_1(\lambda r_2) - I_1(\lambda r_2)\,K_1(\lambda r_1) \right]$$

$$R_0 = \frac{\nu}{2\lambda} \frac{d}{d\lambda} \left(\frac{p}{q} \right) \bigg|_{\lambda=0} = \frac{\nu}{2\lambda} \left(\frac{1}{q} \frac{dp}{d\lambda} - \frac{p}{q^2} \frac{dq}{d\lambda} \right)_{\lambda=0} \tag{A-3}$$

$$\frac{dp}{d\lambda} = \frac{1}{r_1 r_2}\left\{ [r_1^2 K_0(\lambda r_1) - r_2^2 K_0(\lambda r_2)]\ I_1(\lambda r) + r[r_2 K_1(\lambda r_2) - r_1 K_1(\lambda r_1)]\ I_0(\lambda r)\right.$$

$$- \frac{2r_2}{\lambda} K_1(\lambda r_2)\ I_1(\lambda r) + \frac{2r_1}{\lambda} K_1(\lambda r_1)\ I_1(\lambda r) - [r_2^2 I_0(\lambda r_2) - r_1^2 I_0(\lambda r_1)]\ K_1(\lambda r)$$

$$+ r[r_2 I_1(\lambda r_2) - r_1(\lambda r_1)]\ K_0(\lambda r) - \frac{2r_2}{\lambda} I_1(\lambda r_2)\ K_1(\lambda r)$$

$$\left. + \frac{2r_1}{\lambda} I_1(\lambda r_1)\ K_1(\lambda r)\right\}\ e^{\nu\lambda^2 t}$$

$$+ 2\nu\lambda t \left[\left(\frac{K_1(\lambda r_2)}{r_1} - \frac{K_1(\lambda r_1)}{r_2}\right) I_1(\lambda r) - \left(\frac{I_1(\lambda r_2)}{r_1} - \frac{I_1(\lambda r_1)}{r_2}\right) K_1(\lambda r)\right] e^{\nu\lambda^2 t} \tag{A-4}$$

The asymptotic behavior of Eq. (4) for very small values of λ may be demonstrated by first examining the limiting behavior of each separate term. The following asymptotic expressions may be obtained when λ approaches zero:

$$[r_1^2 K_0(\lambda r_1) - r_2^2 K_0(\lambda r_2)]\ I_1(\lambda r) = -\frac{r}{2}\left\{(r_1^2 - r_2^2)\left[\ln\left(\frac{\lambda}{2}\right) + \gamma\right] + r_1^2 \ln r_1 - r_2^2 \ln r_2\right\}\lambda \tag{A-5}$$

$$r[r_2 K_1(\lambda r_2) - r_1 K_1(\gamma r_1)]\ I_0(\gamma r) = \frac{r}{2}\left\{(r_2^2 - r_1^2)\left[\ln\left(\frac{\lambda}{2}\right) + \gamma\right] + r_2^2 \ln r_2 - r_1^2 \ln r_1\right\}\lambda \tag{A-6}$$

$$\frac{2r_1}{\lambda} K_1(\lambda r_1)\ I_1(\lambda r) - \frac{2r_2}{\lambda} K_1(\lambda r_2)\ I_1(\lambda r) = \frac{r}{2}\left\{(r_1^2 - r_2^2)\left[\ln\left(\frac{\lambda}{2}\right) + \gamma\right] + r_1^2 \ln r_1 - r_2^2 \ln r_2\right\}\lambda \tag{A-7}$$

$$-\left[r_2^2 I_0(\lambda r_2) - r_1^2 I_0(\lambda r_1)\right]\left[K_1(\lambda r)\right] K_1(\lambda r) = -\frac{1}{2}\left\{ r(r_2^2 - r_1^2)\left[\ln\left(\frac{\lambda r}{2}\right) + \gamma\right] - (r_2^2 - r_1^2)\frac{2}{\lambda^2 \gamma}\right\}\lambda$$

(A-8)

$$r\left[r_2 I_1(\lambda r_2) - r_1 I_1(\lambda r_1)\right] K_0(\lambda r) = -\frac{1}{2}\left\{ r(r_2^2 - r_1^2)\left[\ln\left(\frac{\lambda r}{2}\right) + \gamma\right]\right\}\lambda \tag{A-9}$$

$$-\left[\frac{2r_1}{\lambda} I_1(\lambda r_1) K_1(\lambda r) - \frac{2r_2}{\lambda} I_1(\lambda r_2) K_1(\lambda r)\right] = -\frac{1}{2}\left\{ r(r_1^2 - r_2^2)\left[\ln\left(\frac{\lambda r}{2}\right) + \gamma\right] - (r_1^2 - r_2^2)\left(\frac{2}{\lambda^2 \gamma}\right)\right\}\lambda$$

(A-10)

$$I_1(\lambda r_1) K_1(\lambda r_2) - I_1(\lambda r_2) K_1(\lambda r_1) = \frac{r_1^2 - r_2^2}{2 r_1 r_2} \tag{A-11}$$

$$\left[\left(\frac{K_1(\lambda r_2)}{r_1} - \frac{K_1(\lambda r_1)}{r_2}\right) I_1(\lambda r) - \left(\frac{I_1(\lambda r_2)}{r_1} - \frac{I_1(\lambda r_1)}{r_2}\right) K_1(\lambda r)\right] = \frac{1}{2 r_1 r_2}\left(\frac{r_1^2 - r_2^2}{r}\right) \tag{A-12}$$

where γ is the Euler's constant. Therefore,

$$\left.\frac{\nu}{2\lambda}\frac{1}{q}\frac{dp}{d\lambda}\right|_{\lambda=0} = \frac{\nu}{2\lambda}\left\{\frac{-r\left[(r_2^2 - r_1^2)\ln r + r_1^2 \ln r_1 - r_2^2 \ln r_2\right]}{(r_1^2 - r_2^2)} + \frac{2\lambda t}{r}\right\}$$

$$= \frac{\nu}{2}\left\{ r \ln r + r\,\frac{r_2^2 \ln r_2 - r_1^2 \ln r_1}{r_1^2 - r_2^2} + \frac{t}{r}\right\} \tag{A-13}$$

Again, since

$$\frac{dq}{d\lambda} = \left\{ -r_2 I_1(\lambda r_1)\, K_0(\lambda r_2) + r_1 I_1(\lambda r_2)\, K_0(\lambda r_1) + r_1 K_1(\lambda r_2)\, I_0(\lambda r_1) \right.$$

$$\left. - r_2 I_0(\lambda r_2)\, K_1(\lambda r_1) + \frac{2}{\lambda} I_1(\lambda r_2)\, K_1(\lambda r_1) - \frac{2}{\lambda} I_1(\lambda r_1)\, K_1(\lambda r_2) \right\}$$

and

$$-r_2 I_1(\lambda r_1)\, K_0(\lambda r_2) + r_1 I_1(\lambda r_2)\, K_0(\lambda r_1) = -\frac{\lambda r_1 r_2}{2} \ln\left(\frac{r_1}{r_2}\right) \tag{A-14}$$

$$r_1 K_1(\lambda r_2)\, I_0(\lambda r_1) - r_2 I_0(\lambda r_2)\, K_1(\lambda r_2) = \frac{r_1 r_2}{2}\left[\ln\left(\frac{\lambda r_2}{2}\right) + \gamma\right]$$

$$-\frac{r_1 r_2}{2}\left[\ln\left(\frac{\lambda r_1}{2}\right) + \gamma\right] - \frac{1}{2\lambda^2}\left[\left(\frac{r_1}{r_2}\right) - \left(\frac{r_2}{r_1}\right)\right] \tag{A-15}$$

$$\frac{2}{\lambda} I_1(\lambda r_2)\, K_1(\lambda r_1) - \frac{2}{\lambda} I_1(\lambda r_1)\, K_1(\lambda r_2) = \frac{r_1 r_2}{2}\left[\ln\left(\frac{\lambda r_1}{2}\right) + \gamma\right]$$

$$-\frac{r_1 r_2}{2}\left[\ln\left(\frac{\lambda r_2}{2}\right) + \gamma\right] - \frac{1}{2\lambda^2}\left[\left(\frac{r_2}{r_1}\right) - \left(\frac{r_1}{r_2}\right)\right] \tag{A-16}$$

Thus,

$$\frac{\nu}{2\lambda}\frac{p}{q^2}\frac{dq}{d\lambda}\bigg|_{\lambda=0} = -\frac{\nu}{2\lambda}\frac{1}{2r_1r_2}\left[\left(\frac{r_1^2-r_2^2}{r}\right)\left(\frac{r_1^2-r_2^2}{2r_1r_2}\right)^{-2}\left(\frac{\lambda r_1r_2}{2}\ln\left(\frac{r_1}{r_2}\right)\right)\right]$$

$$= -\frac{\nu}{r}\,\frac{\dfrac{r_1r_2}{2}\ln\left(\dfrac{r_1}{r_2}\right)}{\dfrac{r_1^2-r_2^2}{r_1r_2}} = -\frac{\nu}{2r}\,r_1^2r_2^2\,\frac{\ln\left(\dfrac{r_1}{r_2}\right)}{r_1^2-r_2^2} \tag{A-17}$$

Consequently,

$$R_0 = \frac{\nu}{2\lambda}\left(\frac{1}{q}\frac{dp}{d\lambda}-\frac{p}{q^2}\frac{dq}{d\lambda}\right)\bigg|_{\lambda=0} = \frac{\nu}{2}\left\{r\ln\left(\frac{r}{r_1}\right)-r\left(1-\frac{r_1^2}{r^2}\right)\frac{\ln\left(\dfrac{r_2}{r_1}\right)}{r_2^2-r_1^2}+\frac{2\nu t}{r}\right\} \tag{A-18}$$

APPENDIX B

Evaluation of Residues of Simple Poles $s = s_n$

Besides $s = 0$, there are infinite numbers of simple poles $s = s_n$, where s_n are the roots of the equations

$$I_1\left(\frac{s_n}{\nu} r_1\right) K_1\left(\frac{s_n}{\nu} r_2\right) - I_1\left(\frac{s_n}{\nu} r_2\right) K_1\left(\frac{s_n}{\nu} r_1\right) = 0 \tag{B-1}$$

Since

$$\frac{1}{2\nu\lambda}\frac{d}{d\lambda}\left[I_1(\lambda r_1)\, K_1\,(\lambda r_2) - I_1(\lambda r_2)\, K_2(\lambda r_1)\right]\Bigg|_{\lambda=\lambda_n}$$

$$= \frac{1}{2\nu\lambda_n}\left\{-r_2 I_1(\lambda_n r_1)\, K_0(\lambda_n r_2) + r_1 I_1(\lambda_n r_2)\, K_0(\lambda_n r_1) + r_1 K_1(\lambda_n r_2)\, I_0(\lambda_n r_1) - r_2 I_0(\lambda_n r_2)\, K_1(\lambda_n r_1)\right\}$$

$$= \frac{1}{2\nu\lambda_n}\left\{r_1 I_1(\lambda_n r_2)\left[K_0(\lambda_n r_1) + \frac{K_1(\lambda_n r_2)}{I_1(\lambda_n r_2)} I_0(\lambda_n r_1)\right] - r_2 I_1(\lambda_n r_1)\left[K_0(\lambda_n r_2) + \frac{K_1(\lambda_n r_1)}{I_1(\lambda_n r_1)} I_0(\lambda_n r_2)\right]\right\}$$

$$= \frac{1}{2\nu\lambda_n}\left\{\frac{I_1(\lambda_n r_2)}{\lambda_n}\frac{1}{I_1(\lambda_n r_1)} - \frac{I_1(\lambda_n r_1)}{\lambda_n}\frac{1}{I_1(\lambda_n r_2)}\right\}$$

$$= \frac{1}{2\nu\lambda_n^2}\left\{\frac{I_1(\lambda_n r_2)}{I_1(\lambda_n r_1)} - \frac{I_1(\lambda_n r_1)}{I_1(\lambda_n r_2)}\right\} = \frac{1}{2\nu\lambda_n^2}\left\{\frac{I_1^2(\lambda_n r_2) - I_1^2(\lambda_n r_1)}{I_1(\lambda_n r_1)\, I_1(\lambda_n r_2)}\right\}$$

$$R_n = 2\nu\lambda_n^2\left[\frac{I_1(\lambda_n r_1)\, I_1(\lambda_n r_2)}{I_1^2(\lambda_n r_2) - I_1^2(\lambda_n r_1)}\right]\left[\left(\frac{K_1(\lambda_n r_2)}{\lambda_n^4 r_1} - \frac{K_1(\lambda_n r_1)}{\lambda_n^4 r_2}\right) I_1(\lambda_n r) - \left(\frac{I_1(\lambda_n r_2)}{\lambda_n^4 r_1} - \frac{I_1(\lambda_n r_1)}{\lambda_n^4 r_2}\right) K_1(\lambda_n r)\right] e^{-\nu\lambda_n^2 t} \tag{B-2}$$

NOMENCLATURE

B_0	longitudinal magnetic field
I	total current per unit length
I_1, K_1, I_0, K_0	modified Bessel functions
J_r	radial current density
J_1, Y_1	Bessel functions
r	radial distance
R	residue
s	variable in the transformed space
t	time
v_θ	azimuthal velocity component
ν	kinematic viscosity
ρ	density of plasma

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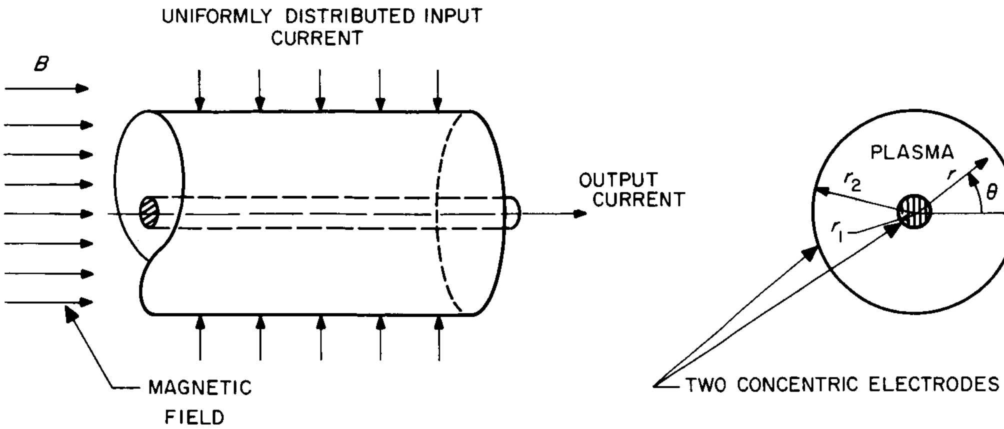

Fig. 1. Model and coordinates

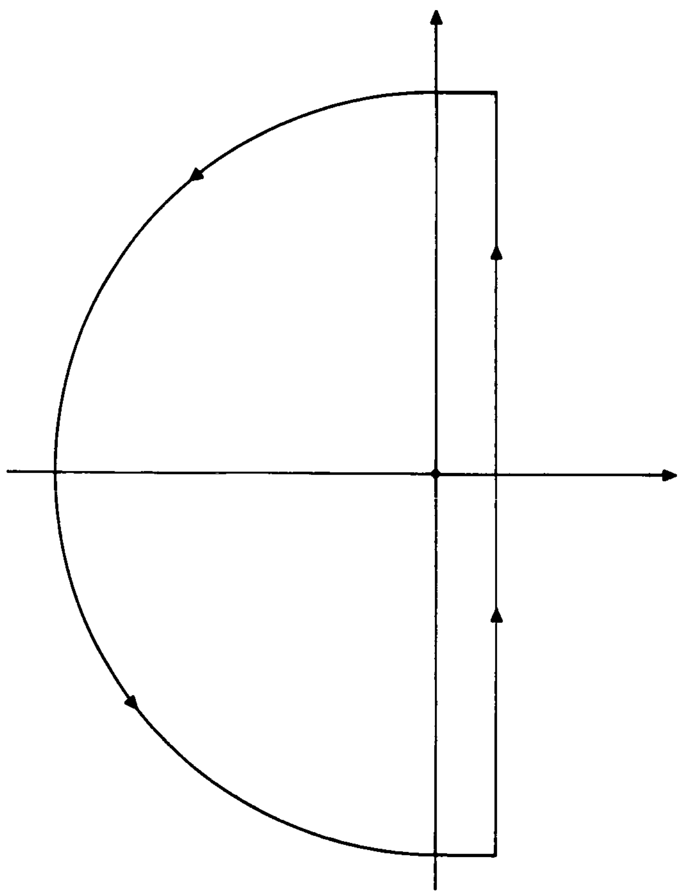

Fig. 2. Contour of integration in s-plane

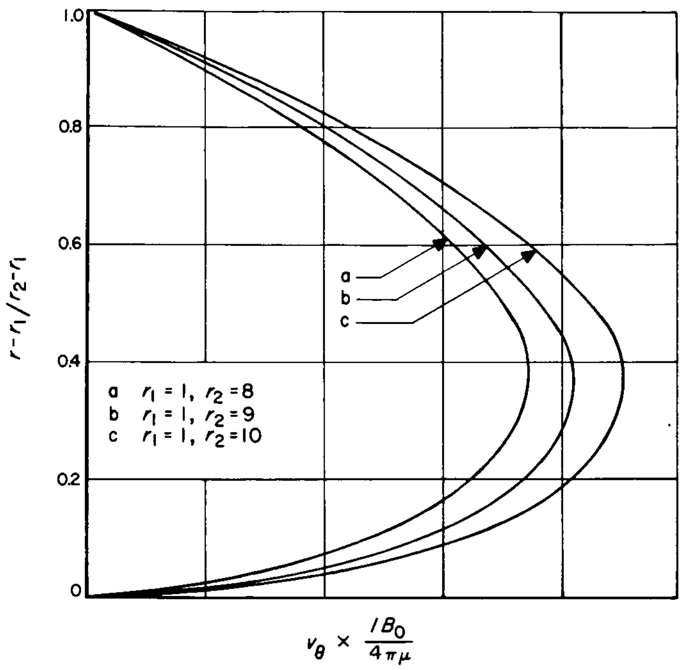

Fig. 3. Asymptotic steady-state velocity profiles

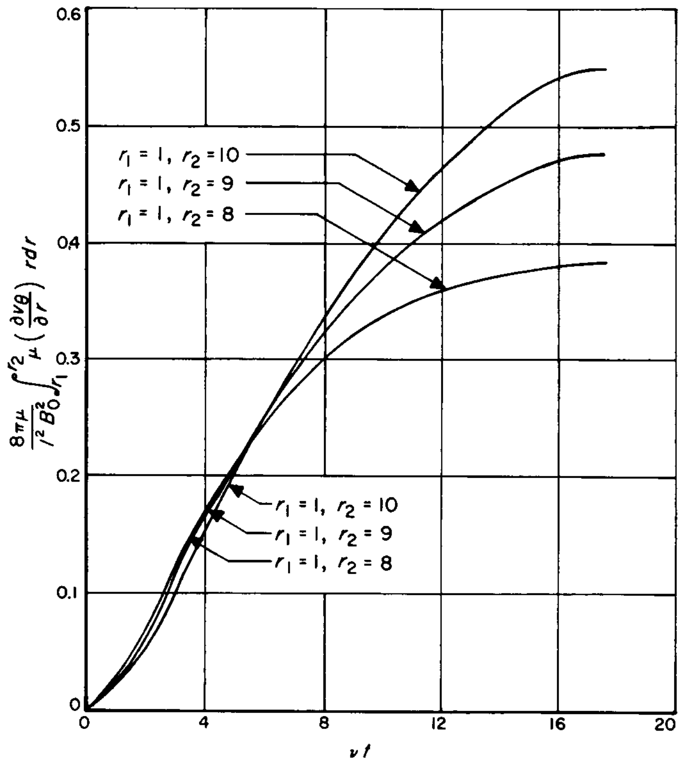

Fig. 4. Variation of total viscous dissipation per unit length of plasma

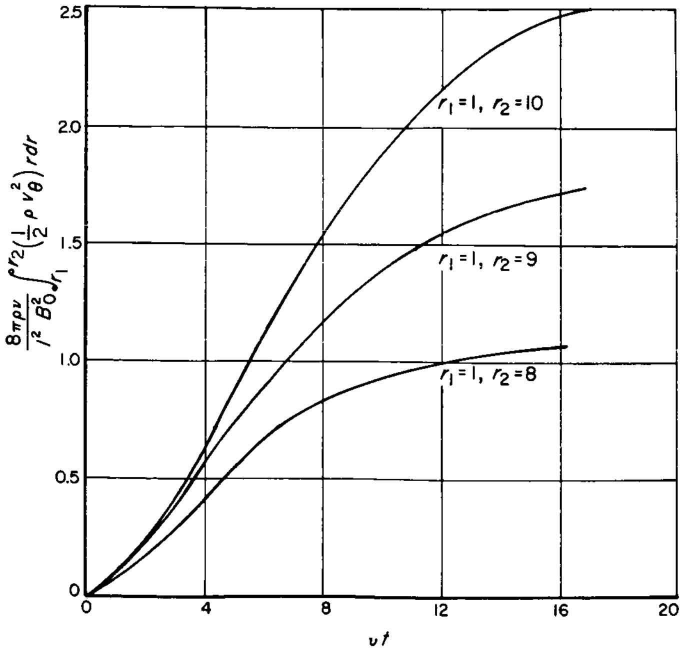

Fig. 5. Variation of total energy of plasma per unit length